Modulated Beams in a Plasma with a Magnetic Field

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Dedicated to Professor HEISENBERG *on the occasion of his sixteeth birthday*

The radiation emitted by a linear density-modulated beam of ions parallel to a steady magnetic field and by a straight beam perpendicular to the field into a non dissipative plasma is investigated. In both cases the energy output becomes infinite at the "*ion*" and "*electron resonance frequencies*".

From the expression for the radiation of the beam which is parallel to the external magnetic field the energy loss of a single particle by ČERENKOV effect is also derived.

The radiation from linear modulated beams of ions into a plasma with an external magnetic field is studied. Some of the results that we derive explicitly here and a discussion of the physical assumptions have been presented in a preceding paper [1]. The connection between the radiation of the beams and the ČERENKOV emission of a single particle is also established.

We describe the beams as given current densities $\boldsymbol{j}_0$ which interact with the plasma according to the non-homogeneous system of equations [2]

$$c \operatorname{curl} \boldsymbol{B} = \dot{\boldsymbol{E}} + 4\pi(\boldsymbol{j} + \boldsymbol{j}_0),$$

$$c \operatorname{curl} \boldsymbol{E} = -\dot{\boldsymbol{B}},$$

$$\varrho \dot{\boldsymbol{v}} = \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B}_0,$$

$$\frac{4\pi}{\omega_p^2} \dot{\boldsymbol{j}} = \boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}_0 + \frac{1}{e} (m_e - m_i) \dot{\boldsymbol{v}},$$

and $$\operatorname{div} \boldsymbol{B} = 0. \qquad (1)$$

$\boldsymbol{B}_0$ is the external magnetic field, e the charge of the proton and the other symbols have the usual meaning. In writing down these equations we have neglected collision and pressure terms.

We consider a beam parallel to $\boldsymbol{B}_0$ and one perpendicular to it. In the first case $\boldsymbol{j}_0$ has the form

$$\boldsymbol{j}_0 = \theta(\varepsilon - r)\,(J_1 + J\, e^{i(m z - \omega t)})\; \boldsymbol{e}_z \qquad (2)$$

in cylindrical coordinates. $\boldsymbol{B}_0 = B_0\, \boldsymbol{e}_z$ and $\theta(r-\varepsilon)$ is 1 if $r < \varepsilon$ and 0 if $r > \varepsilon$. The constants J_1 and J are given by

$$J_1 = I_1/\pi\, \varepsilon^2 \quad \text{and} \quad J = I/\pi\, \varepsilon^2 .$$

$I_1 = N_1\, e\, w$ and $I = N\, e\, w$ $(I_1 \geq I)$ are the unmodulated and modulated current densities respectively, and $w = \omega/m$ is the velocity of the particles of the beam.

The results are given only in the limiting case $\varepsilon = 0$, which corresponds to the beam

$$\boldsymbol{j}_0 = \frac{\delta(r)}{2\pi r}\,(I_1 + I\, e^{i(m z - \omega t)})\, \boldsymbol{e}_z .$$

It is, however, more convenient to use expression (2) in the calculations.

In the second case

$$\boldsymbol{j}_0 = \delta(x)\, \delta(z)\, (I_1 + I\, e^{i(m y - \omega t)})\, \boldsymbol{e}_y \qquad (3)$$

in orthogonal Cartesian coordinates. We don't consider here the curvature of the beam due to the magnetic field but in [3] an example was given where the ion and electron resonances as well as the order of magnitude of the radiation outside the resonances are not modified by the curvature.

The particle moving in the direction of the magnetic field with velocity v is described by the current density

$$\boldsymbol{j}_0 = e\, v\, \frac{\delta(r)}{2\pi r}\, \delta(z - v\, t)\; \boldsymbol{e}_z . \qquad (4)$$

The same coordinate system as for beam (2) is used.

The energy propagation

Some general results will now be given concerning the propagation of the energy in an anisotropic medium, which will be useful for calculating the energy output of beams and particles. If the fields depend on time as $e^{-i\omega t}$ the mean value in time of the energy flux is provided by the real part of the

* by leave from EURATOM.

[1] E. CANOBBIO and R. CROCI, Proc. 5. Int. Conf. on Ionization Phenomena in Gases, Munich 1961 (In Press).

[2] A. SCHLÜTER, Z. Naturforschg. 5 a, 72 [1950].

[3] E. CANOBBIO, Nucl. Fusion 1, 172 [1961].

complex POYNTING vector:

$$\mathbf{S} = \frac{c}{8\pi} \boldsymbol{E} \times \boldsymbol{B}^*$$

where $\boldsymbol{B}^*$ is the complex conjugate of $\boldsymbol{B}$.

It can be shown from the basic equations (1) that this vector satisfies the following equation

$$\operatorname{div} \mathbf{S} = -\frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{j}_0^* \quad (5)$$
$$+ i\omega \left(\frac{1}{8\pi} |\boldsymbol{B}|^2 - \frac{1}{8\pi} |\boldsymbol{E}|^2 + \frac{2\pi}{\omega_p^2} |\boldsymbol{j}|^2 - \frac{\varrho}{2} |\boldsymbol{v}|^2 \right)$$
$$- \frac{(m_i - m_e)}{2 e c \varrho} \boldsymbol{B}_0 \cdot \boldsymbol{j}^* \times \boldsymbol{j} .$$

The physical meaning of the terms on the right side of this equation has already been discussed by KÖRPER [4].

By integrating (5) over a cylindrical volume V of unit height containing the beam, we get for the mean energy flux per unit lenght, W, the expression

$$W = -\frac{1}{2} \operatorname{Re} \left(\int_V \boldsymbol{E} \cdot \boldsymbol{j}_0^* \, dV \right). \quad (6)$$

If the beam has the form (3), this expression reduces to

$$W = -\frac{I}{2} \operatorname{Re} \left((E_y)_{x,\, z=0} \, e^{-i(m y - \omega t)} \right).$$

Due to the presence of the magnetic field $\boldsymbol{B}_0$, there are two waves in the plasma, and the direction of $\operatorname{Re}(\mathbf{S})$, which is that of the propagation of the energy, is not generally that of the waves. For instance, for plane waves with propagation vector $\boldsymbol{n}$ and wave number k, one obtains

$$\operatorname{Re}(\mathbf{S}) = \frac{c^2 k}{8\pi\omega} \left(|\boldsymbol{E}|^2 \boldsymbol{n} - \operatorname{Re}((\boldsymbol{E} \cdot \boldsymbol{n}) \boldsymbol{E}^*) \right). \quad (7)$$

The vector $\operatorname{Re}(\mathbf{S})$ always forms an angle smaller than $\pi/2$ with the propagation vector of the waves, because

$$\operatorname{Re}(\mathbf{S}) \cdot \boldsymbol{n} = \frac{c^2 k}{8\pi\omega} \left(|\boldsymbol{E}|^2 - |\boldsymbol{E} \cdot \boldsymbol{n}|^2 \right) \geq 0 .$$

Equation (6) shows that the mean energy flux is the sum of the contributions of the two waves. Equation (7) allows $\operatorname{Re}(\mathbf{S})$ to have a component directed towards the beam although the phases of the waves propagate from the beam to infinity (see waves 1 in Fig. 1). The calculations show that this is actually so in some frequency intervals. We will see that it is always possible to choose for solutions of equations (1) waves whose phase propagates in such a direction that W is positive (see waves 2 in Fig. 1) as is required by the „Ausstrahlungsbedingungen". If the dispersion relation has two positive roots, both waves transport energy in the radial direction.

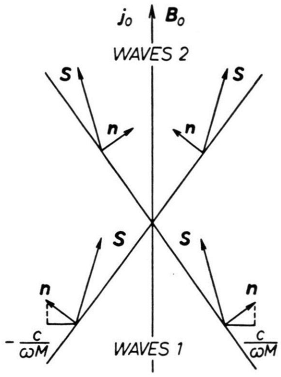

Fig. 1.

Because the two waves are linearly independent solutions of the homogeneous equations outside the beam, they must fulfil the „Ausstrahlungsbedingungen" separately. So we have to choose their direction of propagation in such a way that both make a positive contribution to W. As it can be shown, this is also in agreement with the regularity condition at infinity for a plasma of finite electric conductivity.

[4] K. KÖRPER, Z. Naturforschg. **15 a**, 226 [1960].

Beam parallel to $\boldsymbol{B}_0$

In the cylindrical coordinates previously introduced, we look for solutions of the form

$$\tilde{\boldsymbol{E}} = \boldsymbol{E}(r) \, e^{i(m z - \omega t)}$$

and analogous expressions for the other fields. System (1) then reduces to the following two equations for the quantities $E_\theta(r)$ and $B_\theta(r)$

$$E_\theta'' + \frac{E_\theta'}{r} + \left(\left(\frac{\omega}{c} \right)^2 (n^2 - \gamma) - \frac{1}{r^2} \right) E_\theta = i \left(\frac{\omega}{c} \right)^2 \gamma^{1/2} \alpha B_\theta ,$$

$$B_\theta'' + \frac{B_\theta'}{r} + \left(\left(\frac{\omega}{c} \right)^2 n_1^2 - \frac{1}{r^2} \right) B_\theta = -i \left(\frac{\omega}{c} \right)^2 \gamma^{1/2} \alpha \, n_E^2 E_\theta - \frac{4\pi}{c} J \delta(r - \varepsilon) \quad (8)$$

where $n_1^2 = n_E^2 \left(1 + \gamma \frac{\alpha^2 - 1}{n^2} \right), \quad n_E^2 = 1 - \frac{\sigma}{\varkappa},$

$$n^2 = \frac{(1 - \varkappa + \sigma)^2 - \varkappa/\mu}{(1 - \varkappa)(1 - \varkappa + \sigma) - \varkappa/\mu},$$

$$\alpha = \frac{\sigma \sqrt{\varkappa/\mu}}{(1 - \varkappa)(1 - \varkappa + \sigma) - \varkappa/\mu}, \quad (9)$$

$$\varkappa = \omega^2/\omega_i \omega_e , \quad \sigma = \omega_p^2/\omega_i \omega_e ,$$

$$1/\mu = (\omega_e - \omega_i)^2/\omega_i \omega_e \approx m_i/m_e , \quad \gamma = (c/w)^2 .$$

The quantity $E_z(r)$ which determines the mean energy flux is given by

$$E_z(r) = \frac{i\,c}{\omega\, n_E^2}\left(B_\theta' + \frac{B_\theta'}{r} - \frac{4\pi}{c}\, J\,\theta(\varepsilon - r)\right).$$

The solution of the homogeneous system corresponding to (8) is a linear combination of BESSEL functions of the first order

$$E_\theta = c_1\, Z_1\left(\frac{\omega}{c} M_1\, r\right) + c_2\, \bar{Z}_1\left(\frac{\omega}{c} M_2\, r\right),$$

$$B_\theta = c_1\, b_1\, Z_1\left(\frac{\omega}{c} M_1\, r\right) + c_2\, b_2\, \bar{Z}_1\left(\frac{\omega}{c} M_2\, r\right)$$

where M_1^2 and M_2^2 ($M_2^2 \geq M_1^2$) are the roots of the dispersion relation

$$M^4 + (\gamma - n^2 - n_1^2)\, M^2 + (n^2 - \gamma)\, n_1^2 - \gamma\, \alpha^2\, n_E^2 = 0 \tag{10}$$

and $\qquad b_{1,2} = i/(\sqrt{\gamma}\,\alpha)\,(M_{1,2}^2 + \gamma - n^2)$.

If the regularity condition is satisfied for $r = 0$ and if $M_{1,2}^2 > 0$, the solutions of (8) which behave as outcoming waves when $r > \varepsilon$, are

$$E_\theta = \theta(\varepsilon - r)\left(c_1\, J_1\left(\frac{\omega}{c} M_1\, r\right) + c_2\, J_1\left(\frac{\omega}{c} M_2\, r\right)\right) + \theta(r - \varepsilon)\left(c_3\, H_1^{(1)}\left(\frac{\omega}{c} M_1\, r\right) + c_4\, H_1^{(1)}\left(\frac{\omega}{c} M_2\, r\right)\right),$$

$$\begin{aligned} B_\theta = {} & \theta(\varepsilon - r)\left(c_1\, b_1\, J_1\left(\frac{\omega}{c} M_1\, r\right) + c_2\, b_2\, J_1\left(\frac{\omega}{c} M_2\, r\right)\right) \\ & + \theta(r - \varepsilon)\left(c_3\, b_1\, H_1^{(1)}\left(\frac{\omega}{c} M_1\, r\right) + c_4\, b_2\, H_1^{(1)}\left(\frac{\omega}{c} M_2\, r\right)\right). \end{aligned} \tag{11}$$

$J_1(x)$ and $H_1^{(1)}(x)$ are the BESSEL and HANKEL functions of the first kind respectively and the constants c_i ($i = 1, 2, 3, 4$) are the solutions of

$$c_1\, M_1\, J_1'\left(\frac{\omega}{c} M_1\, \varepsilon\right) + c_2\, M_2\, J_1'\left(\frac{\omega}{c} M_2\, \varepsilon\right) - c_3\, M_1\, H_1^{(1)\prime}\left(\frac{\omega}{c} M_1\, \varepsilon\right) - c_4\, M_2\, H_1^{(1)\prime}\left(\frac{\omega}{c} M_2\, \varepsilon\right) = 0,$$

$$c_1\, b_1\, M_1\, J_1'\left(\frac{\omega}{c} M_1\, \varepsilon\right) + c_2\, b_2\, M_2\, J_1'\left(\frac{\omega}{c} M_2\, \varepsilon\right) - c_3\, b_1\, M_1\, H_1^{(1)\prime}\left(\frac{\omega}{c} M_1\, \varepsilon\right) - c_4\, b_2\, M_2\, H_1^{(1)\prime}\left(\frac{\omega}{c} M_2\, \varepsilon\right) = \frac{4\pi}{\omega}\, J,$$

$$c_1\, J_1\left(\frac{\omega}{c} M_1\, \varepsilon\right) + c_2\, J_1\left(\frac{\omega}{c} M_2\, \varepsilon\right) - c_3\, H_1^{(1)}\left(\frac{\omega}{c} M_1\, \varepsilon\right) - c_4\, H_1^{(1)}\left(\frac{\omega}{c} M_2\, \varepsilon\right) = 0,$$

$$c_1\, b_1\, J_1\left(\frac{\omega}{c} M_1\, \varepsilon\right) + c_2\, b_2\, J_1\left(\frac{\omega}{c} M_2\, \varepsilon\right) - c_3\, b_1\, H_1^{(1)}\left(\frac{\omega}{c} M_1\, \varepsilon\right) - c_4\, b_2\, H_1^{(1)}\left(\frac{\omega}{c} M_2\, \varepsilon\right) = 0.$$

$J_1'(x)$ and $H_1^{(1)\prime}(x)$ denote the derivatives of $J_1(x)$ and of $H_1^{(1)}(x)$ with respect to the argument.

Their expressions are

$$c_1 = -\frac{2\pi^2\gamma^{1/2}\,\alpha\,\varepsilon\, J}{c\,(M_2^2 - M_1^2)}\, H_1^{(1)}\left(\frac{\omega}{c} M_1\, \varepsilon\right),$$

$$c_2 = \frac{2\pi^2\gamma^{1/2}\,\alpha\,\varepsilon\, J}{c\,(M_2^2 - M_1^2)}\, H_1^{(1)}\left(\frac{\omega}{c} M_2\, \varepsilon\right),$$

$$c_3 = -\frac{2\pi^2\gamma^{1/2}\,\alpha\,\varepsilon\, J}{c\,(M_2^2 - M_1^2)}\, J_1\left(\frac{\omega}{c} M_1\, \varepsilon\right),$$

$$c_4 = \frac{2\pi^2\gamma^{1/2}\,\alpha\,\varepsilon\, J}{c\,(M_2^2 - M_1^2)}\, J_1\left(\frac{\omega}{c} M_2\, \varepsilon\right). \tag{12}$$

The fields which behave as incoming waves when $r > \varepsilon$ are obtained by replacing $H_1^{(1)}(x)$ with $H_1^{(2)}(x)$ in (11), and $H_1^{(1)}(x)$ with $-H_1^{(2)}(x)$ in (12) as the calculations show. It can also be proved that if one, or both, $M_{1,2}^2$ are negative, the HANKEL function is replaced by the modified BESSEL function of the third kind, $K_1(x)$, in (11), and by $i\,K_1(x)$ in (12).

From formulae (6) and (4) we get for the mean energy flux

$$W = -\pi\, J\, \mathrm{Re}\left(\int_0^\varepsilon r\, E_z\, \mathrm{d}z\right) = \frac{\pi\, c\, \varepsilon}{\omega\, n_E^2}\, J\, \mathrm{Im}\,(B_\theta(r)_{r=\varepsilon}). \tag{13}$$

When $M_{1,2}^2$ are real negative the solutions of (8) don't contribute to W.

In the frequency intervals where only $M_2^2 > 0$ (see Figs. 2 and 3) for outcoming solutions

$$W = -\frac{2\pi^3\,\varepsilon^2\, J^2}{\omega\, n_E^2\,(M_2^2 - M_1^2)}\,(n^2 - \gamma - M_2^2)\, J_1^2\left(\frac{\omega}{c} M_2\, \varepsilon\right).$$

Going to the limit $\varepsilon = 0$

$$W = \frac{\pi\,\omega\, I^2}{2\, c^2\, n_E^2}\, \frac{M_2^2\,(M_2^2 + \gamma - n^2)}{M_2^2 - M_1^2}.$$

This expression is negative if and only if $n_E^2 < 0$ and $n^2 - \gamma < 0$.

Then incoming waves are to be chosen which simply change the sign of W. When only one wave propagates the energy output is, therefore,

$$W = \frac{\pi\,\omega\,I^2}{2\,c^2\,|\,n_E^2\,|}\;\frac{M_2^2\,|\,M_2^2+\gamma-n^2\,|}{M_2^2-M_1^2}\,.$$

In the intervals where two waves propagate (see Figs. 2 and 3) one incoming wave and one outcoming are to be chosen, in order to have a positive contribution to W from both. With this choice we get

$$W = \frac{\pi\,\omega\,I^2}{2\,c^2\,|\,n_E^2\,|\,(M_2^2-M_1^2)} \cdot (M_1^2\,|\,M_1^2+\gamma-n^2\,|+M_2^2\,|\,M_2^2+\gamma-n^2\,|)\,.$$

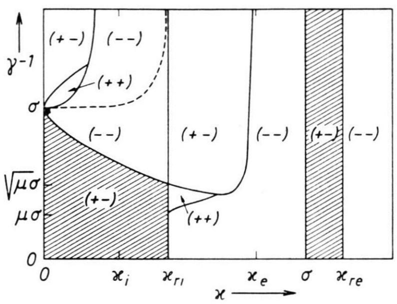

Fig. 2. Qualitative sketch of the solutions of the dispersion relation. The shaded regions are those where outcoming phase waves propagate. The dashed line is the curve $\gamma-1=n^2-1$ ($\varkappa < \varkappa_{ri}$). The ion resonance is possible only on the left of $\varkappa_{ri}$ when $\gamma-1 < -n^4/\alpha^2\,n_E^2 \sim \mu\,\sigma$, and on the right in the other case.

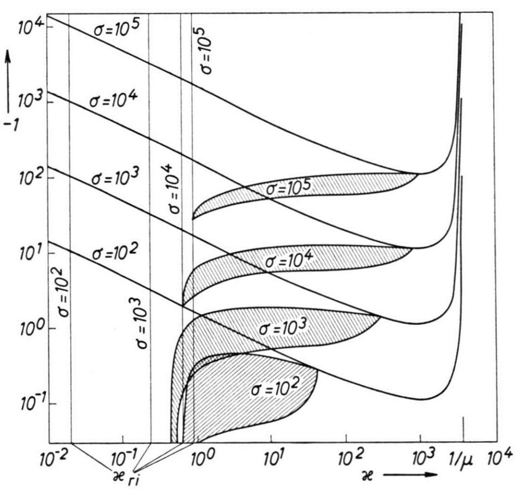

Fig. 3. The curves of fig. 2 are drawn for some values of σ, in the interval $10^{-2} \leq \varkappa \leq 1/\mu$. In the shaded regions $M_{1,2}^2$ are complex.

When the solutions of the dispersion relation (10) are complex, the solutions of (8) which are regular at infinity lead to $W=0$.

The derived expressions for the energy flux can be infinite; this happens only at the infinities of $M_{1,2}^2$, because when n_E^2 goes to zero M_1^2 is negative, and M_2^2 goes to zero as n_E^2. The infinities of $M_{1,2}^2$ occur at the "ion and electron resonance frequencies", ω_{ri} and ω_{re}. They are also the resonance frequencies of n^2 (9), the refractive index for plane waves propagating perpendicular to $\boldsymbol{B}_0$ (ref. [5]), and are the solutions of the equation

$$\Omega = (\omega_i\,\omega_e + \omega_p^2 - \omega^2)\,(\omega_i\,\omega_e - \omega^2) - \omega^2\,(\omega_e - \omega_i)^2 = 0\,.$$

They are given approximately by

$$\omega_{ri}^2 = \frac{\omega_p^2+\omega_i\,\omega_e}{\omega_p^2+\omega_e^2}\,\omega_i\,\omega_e\,, \qquad \omega_{re}^2 = \omega_p^2 + \omega_e^2\,.$$

In the limit $\omega_p \to 0$ they go over into $\omega_{ri}=\omega_i$ and $\omega_{re}=\omega_e$. At the frequencies ω_{ri} and ω_{re}, W is infinite with the same order as n^2. At the ion-resonance the infinite is on the left of ω_{ri} if $\gamma < -n^4/(\alpha^2\,n_E^2)$ and on the right in the other case (see Fig. 2). It is worth noting that if a modulated plane beam travelling parallel to $\boldsymbol{B}_0$ is considered, one finds resonances at the ion and electron resonance frequencies with the same order as n.

Outside the resonances the numerical value of W is extremely small[1] in many practical situations (e. g.: $B \approx 10^3$ Gauss, plasma density $\approx 10^{14}\,\mathrm{cm}^{-3}$, $I \approx 10^{-2}$ A and $\omega \approx \omega_i$).

The single particle

The ČERENKOV radiation of a particle moving parallel or normal to $\boldsymbol{B}_0$ can be easily obtained from the radiation of the corresponding beam. Here we consider explicitly the case of a particle moving parallel to $\boldsymbol{B}_0$. Te other case can be solved analogously.

Let us take the exponential FOURIER time transform of the field quantities. They are defined as

$$\tilde{A}(r,z,\omega) = \int_{-\infty}^{\infty} A(r,z,t)\,e^{-i\,\omega\,t}\,\mathrm{d}t\,.$$

Then $\boldsymbol{j}_0$ goes over into

$$\frac{e\,\delta(r)}{2\,\pi\,r}\,e^{-i\,\omega\,z/v}\,\boldsymbol{e}_z\,.$$

Let us suppose, moreover, that the transformed quantities have the form

$$e^{-i\,\omega\,z/v}\,f(r)\,.$$

In this way from equations (1) we get a set of equa-

[5] K. KÖRPER, Z. Naturforschg. 12 a, 815 [1957].

tions identical with equations (8), provided that m is here replaced by $-\omega/v$, and J by $e/(\pi\,\varepsilon^2)$, and in the limit $\varepsilon=0$.

The fields radiated by the particle are, therefore, proportional to the FOURIER transform of the corresponding beam quantities, provided that when $\omega<0$, the fields are given by the corresponding ones for $\omega>0$ in which $H_1^{(1)}(x)$ $[H_1^{(2)}(x)]$ is replaced by $-H_1^{(2)}(x)$ $[-H_1^{(1)}(x)]$. In this way the same conditions on the energy are satisfied as for $\omega>0$.

The mean energy output per unit time, W_{p}, is obtained from the formula

$$-\operatorname{div}\left(\frac{1}{4\pi}\,\boldsymbol{E}\times\boldsymbol{B}\right)=\boldsymbol{E}\cdot(\boldsymbol{j}+\boldsymbol{j}_0)+\frac{1}{4\pi}\,(\boldsymbol{E}\cdot\dot{\boldsymbol{E}}+\boldsymbol{B}\cdot\dot{\boldsymbol{B}}). \qquad (14)$$

By integrating (14) over the cylindrical volume with limits

$$-\infty<z<\infty\,,\quad 0<\theta\geq 2\pi\,,\quad 0\leq r\leq\varepsilon$$

and $\varepsilon\to 0$, we get

$$W=-e\,v\,\mathrm{Re}\left((E_z)_{\substack{r=0\\ z=v\,t}}\right)=\frac{-e^2\,v}{2\,\pi\,I}\int_{-\infty}^{\infty}\mathrm{Re}\left((E_{zB})_{\substack{r=0\\ m=-\omega/v}}\right)\mathrm{d}\omega\,.$$

E_{zB} is related to the z component of the electric field of the beam which is parallel to $\boldsymbol{B}_0$ by the equation

$$E_z=E_{zB}\,e^{i(m\,z-\omega\,t)}\,.$$

Remembering formula (13), we can write

$$W_{\mathrm{p}}=\frac{e^2\,v}{\pi\,I^2}\int_{-\infty}^{\infty}((W)_{m=-\omega/v})\,\mathrm{d}\omega=\frac{e^2\,v}{2\,c^2}\int_{-\infty}^{\infty}\overline{W}\,\omega\,\mathrm{d}\omega\,.$$

W is the energy output of the beam and $\overline{W}$ is defined by

$$(W)_{m=-\omega/v}=\frac{\pi\,\omega\,I^2}{2\,c^2}\,\overline{W}\,.$$

The mean energy loss per unit path is, therefore,

$$\frac{\mathrm{d}W_{\mathrm{p}}}{\mathrm{d}z}=\frac{e^2}{2\,c^2}\int_{-\infty}^{\infty}\overline{W}\,\omega\,\mathrm{d}\omega\,. \qquad (15)$$

This expression diverges logarithmically at the frequencies ω_{ri} and ω_{re}, but a "cut-off" is provided by a coherence condition on the radiation which determines a lower limit for the wave lengths.

Formula (15) reduces to the expression for the mean energy loss already found in [6] in the limit $\omega_{\mathrm{i}}=0$. In this case, of course, the ion-resonance disappears.

Beam perpendicular to $\boldsymbol{B}_0$

Let us suppose that the fields depend on time as $e^{-i\,\omega\,t}$. Then the electric field is given by the equation

$$\begin{aligned}&\boldsymbol{E}+\frac{c^2}{\omega_{\mathrm{p}}^2}\left(\operatorname{curl}\operatorname{curl}\boldsymbol{E}-\left(\frac{\omega}{c}\right)^2\boldsymbol{E}-\frac{4\,\pi\,i\,\omega}{c^2}\,\boldsymbol{j}_0\right)+\frac{i\,c(m_{\mathrm{i}}-m_{\mathrm{e}})}{4\,\pi\,e\,\varrho\,\omega}\left[\left(\operatorname{curl}\operatorname{curl}\boldsymbol{E}-\left(\frac{\omega}{c}\right)^2\boldsymbol{E}-\frac{4\,\pi\,i\,\omega}{c^2}\,\boldsymbol{j}_0\right)\times\boldsymbol{B}_0\right] \qquad (16)\\&+\frac{1}{4\,\pi\,\varrho\,\omega^2}\left\{\left[\boldsymbol{B}_0\cdot\left(\operatorname{curl}\operatorname{curl}\boldsymbol{E}-\left(\frac{\omega}{c}\right)^2\boldsymbol{E}-\frac{4\,\pi\,i\,\omega}{c^2}\,\boldsymbol{j}_0\right)\right]\boldsymbol{B}_0-|\boldsymbol{B}_0|^2\left(\operatorname{curl}\operatorname{curl}\boldsymbol{E}-\left(\frac{\omega}{c}\right)^2\boldsymbol{E}-\frac{4\,\pi\,i\,\omega}{c^2}\,\boldsymbol{j}_0\right)\right\}=0\,.\end{aligned}$$

Taking now the exponential FOURIER transform on the variables x and z, we have

$$\tilde{\mathbf{E}}(p,q,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{E}(x,z,t)\,e^{i(p\,x+q\,z)}\,\mathrm{d}x\,\mathrm{d}z\,.$$

Dropping the $(\sim)$ for simplicity, equation (16) written in components becomes

$$\begin{aligned}&p\,q\,E_x+m\,q\,E_y+\left(\left(\frac{\omega}{c}\right)^2 n_{\mathrm{E}}^2-m^2-p^2\right)E_z=0\,,\\&\left(1+\left(\frac{c}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,(m^2+q^2)\right)E_x+\left(\alpha+i\left(\frac{c}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,m\,p\right)i\,E_y-\left(\frac{c}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,p\,q\,E_z=0\,, \qquad (17)\\&\left(\alpha-i\left(\frac{c}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,m\,p\right)E_x+\left(1+\left(\frac{-}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,(p^2+q^2)\right)i\,E_y-i\left(\frac{c}{\omega}\right)^2\frac{\alpha^2-1}{n^2}\,m\,q\,E_z=-\frac{4\,\pi}{\omega}\,\frac{\alpha^2-1}{n^2}\,I\,.\end{aligned}$$

Generally we are not able to perform the integrations of the solutions of this system. This is true even near the beam, so that W cannot be obtained. But the problem is solvable at some interesting frequencies

[6] A. A. KOLOMENSKII, Dokl. Akad. Nauk, SSSR **106**, (6), 982 [1956].

and it is also possible to determine the position and order of infinity of the resonances. Integrations are possible in the low frequency region, where displacement currents and electron inertia can be neglected. Then equations (17) can be written

$$E_z = 0,$$

$$\left(1 + \left(\frac{c}{\omega}\right)^2 \frac{\alpha^2 - 1}{n^2} (m^2 + q^2)\right) E_x + \left(\alpha + i \left(\frac{c}{\omega}\right)^2 \frac{\alpha^2 - 1}{n^2} m\, p\right) i\, E_y = 0,$$

$$\left(\alpha - i \left(\frac{c}{\omega}\right)^2 \frac{\alpha^2 - 1}{n^2} m\, p\right) E_x + \left(1 + \left(\frac{c}{\omega}\right)^2 \frac{\alpha^2 - 1}{n^2} (p^2 + q^2)\right) i\, E_y = - \frac{4\pi}{\omega} \frac{\alpha^2 - 1}{n^2} I.$$

To calculate W we need the transform of

$$E_y = \frac{4\pi i \omega I}{c^2} \frac{q^2 - q_1^2 + (\omega/c)^2 \gamma}{q^2 - q_1^2} \frac{1}{p^2 - \dfrac{(q^2 - q_2^2)(q_3^2 - q^2)}{q^2 - q_1^2}}$$

where $\qquad q_1^2 = -(\omega/c)^2 \cdot n^2/(\alpha^2 - 1)$

and $q_{2,3}^2$ are the roots of the equation

$$q^4 + \left(\frac{\omega}{c}\right)^2 \left(\gamma + 2 \frac{n^2}{\alpha^2 - 1}\right) q^2 + \left(\frac{\omega}{c}\right)^4 (\gamma - n^2) \frac{n^2}{\alpha^2 - 1} = 0.$$

We note now that

$$\lim_{\eta \to 0} \int_{-\infty}^{\infty} e^{i(\pm c_1 + i\eta)|x|} e^{ipx}\, dx = \lim_{\eta \to 0} \frac{2(\eta \mp i c_1)}{\eta^2 - (c_1^2 - p^2) \mp 2 i \eta c_1} = \mp \frac{2 i c_1}{p^2 - c_1^2}$$

where η is a real positive number and $c_1^2 > 0$.

We assume that $\mp 2 i c_1/(p^2 - c_1^2)$ is the transform of $e^{\pm i c_1 |x|}$. The real part of $E_y(x, z)_{x,z=0}$ is then obtained by integrating

$$\mp \frac{2\pi\omega I}{c^2} \left(\frac{q^2 - q_1^2 + (\omega/c)^2 \gamma}{c_1 (q^2 - q_1^2)}\right),$$

over the intervals defined by

$$c_1^2 = \frac{(q^2 - q_2^2)(q_3^2 - q^2)}{(q^2 - q_1^2)} > 0,$$

while the imaginary part is obtained by integrating over the other intervals. Outcoming waves correspond to the "minus" sign, incoming waves to the "plus".

The integration over q gives a linear combination of complete elliptic integrals of the first and third kind for $E(x, z)_{x,z=0}$ and, therefore, for W. For example, when $\gamma < \sigma$ and $\omega > \omega_i$ (ref [7])

$$W = (\omega I^2/c^2) \cdot [(q_1^2 - q_2^2)\, q_3^2]^{-1/2} \cdot (q_1^2\, \Pi(\alpha^2, k) + (\omega/c)^2 \gamma\, K(k)),$$

$$k^2 = q_3^2 (q_1^2 - q_2^2)/[q_2^2 (q_1^2 - q_3^2)],$$

$$\alpha^2 = q_3^2/(q_3^2 - q_1^2) \qquad (k^2 < \alpha^2 < 1).$$

One can get an idea of the behaviour of the fields in the space by considering the fields at $\omega = \omega_i$ where one can perform all the calculations and find

$$\mathrm{Re}(E_y(x, z)) = -(\sqrt{2}/c^2)\, \pi \omega I\, \theta(w^2 - v_A^2) \cdot J_0((\omega/\sqrt{2}) \sqrt{[(1/v_A^2) - (1/w^2)](2x^2 + z^2)}).$$

$J_0(x)$ is the Bessel function of the first kind and zeroth order and v_A is the Alfvén velocity. The energy flux is

$$W = \frac{\pi \omega I^2}{\sqrt{2}\, c^2} \theta(w^2 - v_A^2). \tag{18}$$

The expression for $\mathrm{Re}(E_y(x, z)_{x,z=0})$ at $\omega = \omega_i$ can be derived without approximations because equations (17) have a simpler solution which can be integrated. The same is true for $\omega = \omega_e$.

At $\omega = \omega_{i,e}$ $\mathrm{Re}(E_y(x, z)_{x,z=0})$ becomes

$$-\frac{2\omega I}{c^2} \int_{s_2^2}^{s_3^2} \sqrt{\frac{s_1^2 + s^2}{(s^2 - s_2^2)(s_3^2 - s^2)}}\, ds$$

where $\qquad s = c\,p/\omega, \qquad -s_1^2 = n_E^2 - \gamma,$

$$s_2^2 = 2 n_E^2 - \gamma, \qquad s_3^2 = \sigma/(1 - \varkappa) + 2 - \gamma.$$

If $s_2^2 < 0$ the integral should be extended from 0 to s_3^2. We give the explicit result only in the case $\omega_p > \omega_i$ at $\omega = \omega_i$, and $\omega_p > \omega_e$ at $\omega = \omega_e$, which are the most interesting in practice. We get

$$W_{\omega=\omega_i} = \frac{2\omega I^2}{c^2} \frac{s_1^2}{\sqrt{-s_2^2 (s_1^2 + s_3^2)}} \cdot \Pi\left(\frac{s_3^2}{s_1^2 + s_3^2}, \left(\frac{-s_3^2 (s_1^2 + s_2^2)}{s_1^2 (s_2^2 - s_3^2)}\right)^{1/2}\right) \cdot \theta(s_3^2) \tag{19}$$

and $W_{\omega=\omega_e} = 0$ because s_3^2 is then negative.

When $\omega_p < \omega_i$ ($\omega_p < \omega_e$) we get

$$W_{\omega=\omega_{i,e}} = C(\omega_{i,e})\, \Pi(\alpha^2(\omega_{i,e}), k(\omega_{i,e}))$$

where the form of $C(\omega_{i,e})$, $\alpha^2(\omega_{i,e})$ and $k(\omega_{i,e})$ depends on the sign of s_2^2. Formula (18) previously found for the frequency $\omega = \omega_i$ can be obtained from (19) when $\omega_p \gg \omega_i$ because then $s_3^2 \ll |s_{1,2}^2|$ and the integral gives $\pi/\sqrt{2}$.

At the frequency $\omega = \omega_p$ when $\omega_p > \omega_i$ the energy output is zero.

[7] P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals, Springer-Verlag, Berlin 1954.

It will now be proved that the energy flux of the beam perpendicular to $\boldsymbol{B}_0$ is infinite only at the frequencies ω_{ri} and ω_{re} and that it has the same order of infinity as n.

That is the same resonances which have been found in the plane beams [3, 7], when the particles travel perpendicular to the magnetic field with an appropriate value of the velocity determined by the frequency and the plasma properties do not occur.

The general expression to be inverted may be written in the form

$$E_y(p,q) = \frac{4\pi i\,\omega\,\omega_{\text{p}}^2 I}{c^2\,\Omega}\left(\frac{c_1}{p^2-\varepsilon^2}+\frac{c_2}{p^2-\delta^2}\right)$$

where ε^2 and δ^2 are the roots of

$$u^2 + a_3\,u + a_4 = 0 \qquad \text{and}$$

$$c_1 = \frac{a_1\,\varepsilon^2 + a_2}{\varepsilon^2-\delta^2}, \qquad c_2 = -\frac{a_1\,\delta^2+a_2}{\varepsilon^2-\delta^2}.$$

a_1, a_2, a_3 and a_4 are polynomials in q^2 of the order 0, 1, 1 and 2, respectively.

After the first integration we obtain

$$E_y(x,z) = \frac{\omega\,\omega_{\text{p}}^2 I}{c^2\,\Omega}\left(\pm\int\frac{c_1}{\varepsilon}\exp(\mp i\,\varepsilon\,|x| - i\,q\,z)\,\mathrm{d}q \pm\int\frac{c_2}{\delta}\exp(\mp i\,\delta\,|x|-i\,q\,z)\,\mathrm{d}q\right)$$

where the intervals of integration are defined by $\varepsilon^2>0$ and $\delta^2>0$. The sign has to be chosen so that both integrals are negative at $(x=0,\ z=0)$ in order to satisfy the condition on the energy cited above.

One sees that the integral over q is finite. The only infinities in the energy flux arise at the frequencies where $\Omega=0$, that is, at $\omega=\omega_{\text{ri}}$ and $\omega=\omega_{\text{re}}$. Differently from the parallel beam, the ion-resonance is possible on both sides of ω_{ri}, for whatever w. In fact, on the left of the resonance frequency ω_{ri} the integral is approximately

$$\operatorname{Re}(E_y(x,z)) \propto |n|\operatorname{Re}\left(\int_0^Q \frac{\exp\{i(-|\alpha\,n_{\text{E}}/n|\,\sqrt{Q-q^2}\,|x|-q\,z\}}{\sqrt{Q-q^2}}\,dq\right)$$

where $\qquad Q = -\,[\omega\,n^2/(c\,\alpha\,n_{\text{E}})]^2$

and on the right

$$\propto |n|\operatorname{Re}\left(\int_Q^\infty \frac{\exp\{i(+|\alpha\,n_{\text{E}}/n|\,\sqrt{q^2-Q^2}\,|x|\,q\,z\}}{\sqrt{q^2-Q}}\,dq\right).$$

Both are infinite with the same order as n.

On the left of ω_{re} the expression to be calculated is

$$\operatorname{Re}(E_y(x,z)) \propto |n|\operatorname{Re}\left(\int_0^\infty \frac{\exp\{i(-|\alpha\,n_{\text{E}}/n|\,\sqrt{q^2-Q}\,|x|\,q\,z\}}{\sqrt{q^2-Q}}\,dq\right)$$

which is infinite as n. On the other side the energy flux is zero because $E_y(x,z)_{x,z=0}$ is pure imaginary.

In plane beams propagating perpendicularly to the magnetic field the situation is the following: if the beam is parallel to $\boldsymbol{B}_0$, the energy flux is infinite at the frequencies ω_{ri} and ω_{re} with the same order as n. The flux is also infinite when

$$w = c/n. \tag{20}$$

If the beam is perpendicular to $\boldsymbol{B}_0$ it can be shown that the resonance occurs only when

$$w = c/\sqrt{(1-\varkappa)\,n^2+\varkappa}. \tag{21}$$

In the limit of very low frequencies equations (20) and (21) give $w = v_{\text{A}}$ (ref. [8]).

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[8] R. Kippenhahn and H. L. de Vries, Z. Naturforschg. **15 a**, 506 [1960].